Assignment 1

Tales Padilha
tales.padilha@economics.ox.ac.uk

Department of Economics
University of Oxford

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Exercise 7.2

The APARCH(1,1,1) model is defined as:

$$
\begin{aligned}
r_t &= \mu_t + \epsilon_t \\
\sigma_t^\delta &= \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta \\
\epsilon_t &= \sigma_t e_t \\
e_t &\overset{i.i.d.}{\sim} N(0,1)
\end{aligned}
$$

We are asked to show the relationship between this model and:

- ARCH(1)
- GARCH(1,1)
- AVGARCH(1,1)
- TARCH(1,1,1)
- GJR-GARCH(1,1,1)

Exercise 7.2

The only difference in all the models that we are analysing is the process that we assume for the "variance".

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For the ARCH(1):

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$

For the GARCH(1, 1):

Exercise 7.2

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For the ARCH(1):

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$

For the GARCH(1, 1):

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

Exercise 7.2

For the AVGARCH(1,1):

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$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \beta\sigma_{t-1}$$

For the TARCH(1,1,1):

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For the AVGARCH(1,1):

$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \beta\sigma_{t-1}$$

For the TARCH(1,1,1):

$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \gamma|\epsilon_{t-1}|\mathbb{1}_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}$$

For the GJR-GARCH(1,1,1):

Exercise 7.2

For the AVGARCH(1,1):

$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \beta\sigma_{t-1}$$

For the TARCH(1,1,1):

$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \gamma|\epsilon_{t-1}|\mathbb{1}_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}$$

For the GJR-GARCH(1,1,1):

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma\epsilon_{t-1}^2\mathbb{1}_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}^2$$

Exercise 7.2.1 ARCH(1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Exercise 7.2.1 ARCH(1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Therefore if we set:

- $\beta = \gamma = 0$
- $\delta = 2$

Since $|\epsilon_{t-1}|^2 = \epsilon_{t-1}^2$ we get:

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2$$

Which is the variance process in the ARCH(1).

Exercise 7.2.2 GARCH(1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Exercise 7.2.2 GARCH(1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Therefore if we set:

- $\gamma = 0$
- $\delta = 2$

Since $|\epsilon_{t-1}|^2 = \epsilon_{t-1}^2$ we get:

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

Which is the variance process in the GARCH(1,1).

Exercise 7.2.3 AVGARCH(1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Exercise 7.2.3 AVGARCH(1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Therefore if we set:

- $\gamma = 0$
- $\delta = 1$

We get:

$$\sigma_t = \omega + \alpha|\epsilon_{t-1}| + \beta\sigma_{t-1}$$

Which is the standard deviation process in the AVGARCH(1,1).

Exercise 7.2.4 TARCH(1,1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Exercise 7.2.4 TARCH(1,1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Let's start by setting $\delta = 1$:

$$\sigma_t = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1}) + \beta\sigma_{t-1}$$

Since the TARCH(1,1,1) has the indicator function on $\epsilon_{t-1} < 0$, let's think of the two possibilities.

If $\epsilon_{t-1} \geq 0$ (i.e. $\epsilon_{t-1} = |\epsilon_{t-1}|$):

$$\begin{aligned}
\sigma_t &= \omega + \alpha(|\epsilon_{t-1}| + \gamma|\epsilon_{t-1}|) + \beta\sigma_{t-1} \\
&= \omega + \alpha|\epsilon_{t-1}| + \alpha\gamma|\epsilon_{t-1}| + \beta\sigma_{t-1} \\
&= \omega + (\alpha + \alpha\gamma)|\epsilon_{t-1}| + \beta\sigma_{t-1}
\end{aligned}$$

Exercise 7.2.4 TARCH(1,1,1)

On the other hand, if $\epsilon_{t-1} < 0$ (i.e. $\epsilon_{t-1} = -|\epsilon_{t-1}|$):

$$\begin{aligned}\sigma_t &= \omega + \alpha[|\epsilon_{t-1}| + \gamma(-|\epsilon_{t-1}|)] + \beta\sigma_{t-1} \\ &= \omega + \alpha|\epsilon_{t-1}| - \alpha\gamma|\epsilon_{t-1}| + \beta\sigma_{t-1} \\ &= \omega + (\alpha - \alpha\gamma)|\epsilon_{t-1}| + \beta\sigma_{t-1}\end{aligned}$$

Exercise 7.2.4 TARCH(1,1,1)

To summarize:

- If $\epsilon_{t-1} \geq 0$: $\sigma_t = \omega + (\alpha + \alpha\gamma)|\epsilon_{t-1}| + \beta\sigma_{t-1}$
- If $\epsilon_{t-1} < 0$: $\sigma_t = \omega + (\alpha - \alpha\gamma)|\epsilon_{t-1}| + \beta\sigma_{t-1}$

We can write this process for the standard deviation as:

$$\sigma_t = \omega + \tilde{\alpha}|\epsilon_{t-1}| + \tilde{\gamma}|\epsilon_{t-1}|\mathbb{1}_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}$$

where $\tilde{\alpha} = \alpha + \alpha\gamma$ and $\tilde{\gamma} = -2\alpha\gamma$.

Which is the standard deviation process in the TARCH(1,1,1).

Exercise 7.2.5 GJR-GARCH(1,1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Exercise 7.2.5 GJR-GARCH(1,1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Let's start by setting $\delta = 2$:

$$\sigma_t^2 = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^2 + \beta\sigma_{t-1}^2$$

Since the GJR-GARCH(1,1,1) has the indicator function in $\epsilon_{t-1} < 0$, let's think of the two possibilities.

Exercise 7.2.5 GJR-GARCH(1,1,1)

The "variance" process in the APARCH(1,1,1) is defined as:

$$\sigma_t^\delta = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta$$

Let's start by setting $\delta = 2$:

$$\sigma_t^2 = \omega + \alpha(|\epsilon_{t-1}| + \gamma\epsilon_{t-1})^2 + \beta\sigma_{t-1}^2$$

Since the GJR-GARCH(1,1,1) has the indicator function in $\epsilon_{t-1} < 0$, let's think of the two possibilities.

If $\epsilon_{t-1} \geq 0$ (i.e. $\epsilon_{t-1} = |\epsilon_{t-1}|$):

$$\begin{aligned}
\sigma_t^2 &= \omega + \alpha(|\epsilon_{t-1}| + \gamma|\epsilon_{t-1}|)^2 + \beta\sigma_{t-1}^2 \\
&= \omega + \alpha(|\epsilon_{t-1}|^2 + 2\gamma|\epsilon_{t-1}||\epsilon_{t-1}| + \gamma^2|\epsilon_{t-1}|^2) + \beta\sigma_{t-1}^2 \\
&= \omega + \alpha(|\epsilon_{t-1}|^2 + 2\gamma|\epsilon_{t-1}|^2 + \gamma^2|\epsilon_{t-1}|^2) + \beta\sigma_{t-1}^2 \\
&= \omega + (\alpha + 2\alpha\gamma + \alpha\gamma^2)|\epsilon_{t-1}|^2 + \beta\sigma_{t-1}^2 \\
&= \omega + (\alpha + 2\alpha\gamma + \alpha\gamma^2)\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2
\end{aligned}$$

Exercise 7.2.5 GJR-GARCH(1,1)

On the other hand, if $\epsilon_{t-1} < 0$ (i.e. $\epsilon_{t-1} = -|\epsilon_{t-1}|$):

$$
\begin{aligned}
\sigma_t^2 &= \omega + \alpha[|\epsilon_{t-1}| + \gamma(-|\epsilon_{t-1}|)]^2 + \beta\sigma_{t-1}^2 \\
&= \omega + \alpha(|\epsilon_{t-1}| - \gamma|\epsilon_{t-1}|)^2 + \beta\sigma_{t-1}^2 \\
&= \omega + \alpha(|\epsilon_{t-1}|^2 - 2\gamma|\epsilon_{t-1}||\epsilon_{t-1}| + \gamma^2|\epsilon_{t-1}|^2) + \beta\sigma_{t-1}^2 \\
&= \omega + \alpha(|\epsilon_{t-1}|^2 - 2\gamma|\epsilon_{t-1}|^2 + \gamma^2|\epsilon_{t-1}|^2) + \beta\sigma_{t-1}^2 \\
&= \omega + (\alpha - 2\alpha\gamma + \alpha\gamma^2)|\epsilon_{t-1}|^2 + \beta\sigma_{t-1}^2 \\
&= \omega + (\alpha - 2\alpha\gamma + \alpha\gamma^2)\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2
\end{aligned}
$$

Exercise 7.2.5 GJR-GARCH(1,1,1)

To summarize:

- If $\epsilon_{t-1} \geq 0$: $\sigma_t^2 = \omega + (\alpha + 2\alpha\gamma + \alpha\gamma^2)\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$
- If $\epsilon_{t-1} < 0$: $\sigma_t^2 = \omega + (\alpha - 2\alpha\gamma + \alpha\gamma^2)\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

We can write this process for the variance as:

$$\sigma_t^2 = \omega + \tilde{\alpha}\epsilon_{t-1}^2 + \tilde{\gamma}\epsilon_{t-1}^2 \mathbb{1}_{[\epsilon_{t-1}<0]} + \beta\sigma_{t-1}$$

where $\tilde{\alpha} = \alpha + 2\alpha\gamma + \alpha\gamma^2$ and $\tilde{\gamma} = -4\alpha\gamma$.

Which is the variance process in the GJR-GARCH(1,1,1).

Exercise 7.4

We are told that r_t follows the GARCH process:

$$
\begin{aligned}
r_t &= e_t \sigma_t \\
\sigma_t^2 &= \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 \\
e_t &\overset{i.i.d.}{\sim} N(0,1)
\end{aligned}
$$

Let's now check the properties of r_t!

Exercise 7.4.1.a.

We are asked to derive the unconditional expectation $E[r_{t+1}]$:

$$
\begin{aligned}
E[r_{t+1}] &= E[e_{t+1}\sigma_{t+1}] \\
&= E[E_t[e_{t+1}\sigma_{t+1}]] \qquad \text{by L.I.E} \\
&= E[E_t[e_{t+1}]\sigma_{t+1}] \\
&= E[(0)\sigma_{t+1}] \\
&= 0
\end{aligned}
$$

Exercise 7.4.1.b.

We are asked to derive the conditional expectation $E_t[r_{t+1}]$:

$$\begin{aligned} E_t[r_{t+1}] &= E_t[e_{t+1}\sigma_{t+1}] \\ &= E_t[e_{t+1}]\sigma_{t+1} \\ &= (0)\sigma_{t+1} \\ &= 0 \end{aligned}$$

Exercise 7.4.1.c.

We are asked to derive the unconditional variance $VAR[r_{t+1}]$:

$$\begin{aligned}
VAR[r_{t+1}] &= E[(r_{t+1} - E[r_{t+1}])^2] \\
&= E[(r_{t+1} - 0)^2] \qquad \text{from (a)} \\
&= E[r_{t+1}^2] \\
&= E[e_{t+1}^2 \sigma_{t+1}^2] \\
&= E[E_t[e_{t+1}^2 \sigma_{t+1}^2]] \qquad \text{by L.I.E} \\
&= E[E_t[e_{t+1}^2] \sigma_{t+1}^2] \\
&= E[(1)\sigma_{t+1}^2] \\
&= E[\sigma_{t+1}^2] \\
&= \bar{\sigma}^2
\end{aligned}$$

Exercise 7.4.1.c.

But what is the unconditional variance $\bar{\sigma}^2$?

$$\bar{\sigma}^2 = E[\sigma_{t+1}^2]$$
$$\bar{\sigma}^2 = E[\omega + \alpha r_t^2 + \beta \sigma_t^2]$$
$$\bar{\sigma}^2 = \omega + \alpha E[r_t^2] + \beta E[\sigma_t^2]$$

We know that $VAR[r_t] = E[r_t^2] = E[\sigma_t^2] = \bar{\sigma}^2$. Hence:

$$\bar{\sigma}^2 = \omega + \alpha E[r_t^2] + \beta E[\sigma_t^2]$$
$$\bar{\sigma}^2 = \omega + \alpha \bar{\sigma}^2 + \beta \bar{\sigma}^2$$
$$\bar{\sigma}^2 - \alpha \bar{\sigma}^2 - \beta \bar{\sigma}^2 = \omega$$
$$\bar{\sigma}^2 = \frac{\omega}{1 - \alpha - \beta}$$

Exercise 7.4.1.d.

We are asked to derive the conditional variance $VAR_t[r_{t+1}]$:

$$\begin{aligned}
VAR_t[r_{t+1}] &= E_t[(r_{t+1} - E_t[r_{t+1}])^2] \\
&= E_t[(r_{t+1} - 0)^2] \qquad \text{from (b)} \\
&= E_t[r_{t+1}^2] \\
&= E_t[e_{t+1}^2 \sigma_{t+1}^2] \\
&= E_t[e_{t+1}^2]\sigma_{t+1}^2 \\
&= (1)\sigma_{t+1}^2 \\
&= \sigma_{t+1}^2
\end{aligned}$$

Exercise 7.4.1.e.

We are asked to derive the first order autocorrelation of r_t ρ_1:

$$
\begin{aligned}
\rho_1 &= \frac{E[r_t r_{t-1}]}{VAR[r_t]} \\
&= \frac{E[(e_t\sigma_t)(e_{t-1}\sigma_{t-1})]}{VAR[r_t]} \\
&= \frac{E[E_{t-1}[e_t e_{t-1}\sigma_t\sigma_{t-1}]]}{VAR[r_t]} && \text{by L.I.E} \\
&= \frac{E[E_{t-1}[e_t]e_{t-1}\sigma_t\sigma_{t-1}]}{VAR[r_t]} \\
&= \frac{E[(0)e_{t-1}\sigma_t\sigma_{t-1}]}{VAR[r_t]} && \text{from (b)} \\
&= 0
\end{aligned}
$$

Exercise 7.4.3

The final question asks us to describe the h-step ahead forecast for this GARCH(1,1) model for the variance process.

Let's start with a 1-step forecast:

$$\begin{aligned} E_t[\sigma_{t+1}^2] &= E_t[\omega + \alpha r_t^2 + \beta \sigma_t^2] \\ &= \omega + \alpha E_t[r_t]^2 + \beta E_t[\sigma_t^2] \\ &= \omega + \alpha r_t^2 + \beta \sigma_t^2 \end{aligned}$$

Exercise 7.4.3

Now a 2-step forecast:

$$\begin{aligned}
E_t[\sigma^2_{t+2}] &= E_t[\omega + \alpha r^2_{t+1} + \beta\sigma^2_{t+1}] \\
&= \omega + \alpha E_t[r^2_{t+1}] + \beta E_t[\sigma^2_{t+1}] \\
&= \omega + \alpha E_t[e^2_{t+1}\sigma^2_{t+1}] + \beta E_t[\sigma^2_{t+1}] \\
&= \omega + \alpha E_t[e^2_{t+1}]\sigma^2_{t+1} + \beta\sigma^2_{t+1} \\
&= \omega + \alpha\sigma^2_{t+1} + \beta\sigma^2_{t+1} \\
&= \omega + (\alpha + \beta)\,\sigma^2_{t+1}
\end{aligned}$$

Substituting σ^2_{t+1}, which is known at time t, will produce:

$$\begin{aligned}
E_t[\sigma^2_{t+2}] &= \omega + (\alpha + \beta)\left(\omega + \alpha r^2_t + \beta\sigma^2_t\right) \\
&= \omega + (\alpha + \beta)\,\omega + (\alpha + \beta)\left(\alpha r^2_t + \beta\sigma^2_t\right)
\end{aligned}$$

Exercise 7.4.3

Finally a 3-step forecast:

$$
\begin{aligned}
E_t[\sigma^2_{t+3}] &= E_t[\omega + \alpha r^2_{t+2} + \beta\sigma^2_{t+2}] \\
&= \omega + \alpha E_t[r^2_{t+2}] + \beta E_t[\sigma^2_{t+2}] \\
&= \omega + \alpha E_t[e^2_{t+2}\sigma^2_{t+2}] + \beta E_t[\sigma^2_{t+2}] \\
&= \omega + \alpha E_t[E_{t+1}[e^2_{t+2}\sigma^2_{t+2}]] + \beta E_t[\sigma^2_{t+2}] \qquad \text{by L.I.E} \\
&= \omega + \alpha E_t[E_{t+1}[e^2_{t+2}]\sigma^2_{t+2}] + \beta E_t[\sigma^2_{t+2}] \\
&= \omega + \alpha E_t[(1)\sigma^2_{t+2}] + \beta E_t[\sigma^2_{t+2}] \\
&= \omega + (\alpha + \beta) E_t[\sigma^2_{t+2}]
\end{aligned}
$$

Using our $E_t[\sigma^2_{t+2}]$ from the previous step:

$$
\begin{aligned}
E_t[\sigma^2_{t+3}] &= \omega + (\alpha + \beta) E_t[\sigma^2_{t+2}] \\
&= \omega + (\alpha + \beta)[\omega + (\alpha + \beta)\,\omega + (\alpha + \beta)\,(\alpha r^2_t + \beta\sigma^2_t)] \\
&= \omega + (\alpha + \beta)\,\omega + (\alpha + \beta)^2\,\omega + (\alpha + \beta)^2\,(\alpha r^2_t + \beta\sigma^2_t)
\end{aligned}
$$

Exercise 7.4.3

By looking at the previous forecasts, we can see that a pattern emerges.

More generally, we can use this pattern to write:

$$E_t[\sigma^2_{t+h}] = \sum_{i=0}^{h-1} (\alpha + \beta)^i \omega + (\alpha + \beta)^{h-1} \left(\alpha r_t^2 + \beta \sigma_t^2\right)$$

The h-step ahead forecast is an exponentially declining function of the time $t + 1$ forecast plus a constant.

For large h, the forecast converges to $\bar{\sigma}^2$.